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Algebras for causal external electromagnetic interaction in higher-spin theories

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Abstract. Using the theory of Young symmetrizers we show how to obtain algebras which are sufficient for causal propagation of higher-spin particles interacting with an external minimal electromagnetic field. The commutation relations of the algebra are derived already expressed in irreducible Young symmetrizer form. An example is given which is not equivalent to any known causal theory and it is shown that the algebra is infinite. Thus, the requirement of causal electromagnetic interaction is not sufficient to generate a finite algebra and non-trivial sub-algebras may exist which are causal.

1. Introduction

There is now considerable literature on the Velo–Zwanzinger inconsistency for the external field problem in higher-spin theories (Velo and Zwanzinger 1969), in which apparently manifestly covariant field equations have solutions which propagate acausally. It is still true that no causal theories are known for spin greater than one interacting with an external field, even at the non-quantized level. Nor is the problem removed by consideration of full interaction, as opposed to external field interaction (Capri and Shamaly 1974) or by quantization (Johnson and Sudarshan 1961).

The usual approach to the external field problem for higher spin is to consider specific constructed free field theories, in which the field variables have known transformation properties and the field equation can be written explicitly in tensor–spinor form or matrix form. Thus, starting with a known free field theory one introduces the external interaction by standard recipes and investigates the propagation properties of the resulting theory by the standard methods of analysis of differential equations (Courant and Hilbert 1962, Hormander 1963). There are many examples of this approach in the literature and the basic conclusion is that whilst all known spin-1 theories are causal when interacting minimally with an external electromagnetic field, and also with some other fields, all known higher-spin theories either propagate acausally or not at all, in any external field (Capri and Shamaly 1972, Velo 1972, Singh 1973). There are exceptions in the case of theories with mass–spin spectra, but these usually exhibit other problems and we do not consider them here (Cox 1976). It is tempting to conclude that all higher-spin theories are causal in interaction, but as there are so few good free field theories which are amenable to simple analysis in the presence of an external field, there is little basis for doing this. In fact the constructed theories for spin greater than one are very limited and all but the simplest (e.g. spin $\frac{3}{2}$ and some spin 2) are too complicated to analyse in the usual way, the main problem being the constraints used to ensure manifest covariance of the field equations.

Another approach to higher-spin field theories is that based on the first-order free field equation:

$$(\Gamma_\mu p_\mu - m)\psi = (\Gamma . p - m)\psi = 0 \tag{1.1}$$

and which studies the algebra of the matrices Γ_μ and m. Here, we take m to be a scalar matrix. In this approach, pioneered by Harish-Chandra (1947a, b) and Bhabha (1949), the enveloping algebra of the Γ_μ is specified *a priori* by imposing certain algebraic relations in these matrices. By standard algebraic methods it is then possible to determine the matrix representations of the Γ_μ and the corresponding mass–spin spectra carried by the field ψ. The prototype of this approach is the well known Duffin–Kemmer theory (Kemmer 1939). For many years this approach was neglected because of the difficulties of finding and manipulating suitable Γ_μ algebras. The complete analysis of the constructive aspect of (1.1) was given by Gel'fand and Yaglom (Gel'fand *et al* 1963) including the infinite-dimensional case, but no attempt was made at a complete algebraic analysis. Shelepin (1960) made such a study and anticipated much of the more recent work of Glass (1971) who made a detailed investigation of the algebraic approach and its relation to known constructed theories. While the algebraic approach is difficult in the case of higher-spin theories, it does have the advantage of generality and allows us to isolate all of the behaviour of the system in the properties of the Γ_μ matrices. In any case there remain very few constructed theories which are not too complicated for interaction analysis.

In this paper we obtain sufficient algebraic conditions for a causal theory based on (1.1) interacting minimally with an external electromagnetic field. The Γ_μ generate a tensor algebra and we obtain the equations in forms irreducible under general linear transformations in the tensor space of Γ_μ products, using standard theory of Young tableaux. The equations are obtained by requiring that the principal part of the true equation of motion for the field ψ (i.e. after constraints are eliminated), derived from (1.1), is Klein–Gordon in the presence of an external field. If this is the case then the theory will clearly be causal. Further, we show that the algebra generated by these conditions is infinite. It is well known that for high spin the usual free field requirements of covariance, unique mass, etc are not sufficient to ensure that the algebra generated by the Γ_μ is finite and that further conditions are required to make it finite (Harish-Chandra 1947a, b). It is therefore interesting to see that the added condition of causal propagation, in the particular cases considered here, is still not sufficient to guarantee finiteness of the algebra and that we are still at liberty to prescribe further relations. This in itself does not mean that the algebra has non-trivial representations yielding a good causal theory, and in fact it is very difficult to decide this, owing to the complexity of the algebra. However, it is suggestive and provides some motivation for further study of the algebra.

2. Sufficient algebraic conditions for a causal theory

Following Velo and Zwanzinger (1971), we introduce an external electromagnetic field into the theory (1.1) by the minimal replacement:

$$p_\mu \rightarrow \pi_\mu = p_\mu + eA_\mu \tag{2.1}$$

where

$$[\pi_\mu, \pi_\nu]\psi = ieF_{\mu\nu}\psi \tag{2.2}$$

for an arbitrary space–time function ψ.

(1.1) becomes

$$(\Gamma\,.\,\pi - m)\psi = 0. \tag{2.3}$$

In general the time coefficient matrix Γ_4 is singular and so (2.3) will not be a true equation of motion—certain of the field components will not have their time derivatives determined. To obtain from (2.3) a true equation of motion we use the Klein–Gordon divisor $d(p)$, which is a matrix differential operator such that

$$d(p)(\Gamma\,.\,p - m) = (p^2 - m^2)I. \tag{2.4}$$

In the free field case this is easily calculated as a polynomial in p and Γ_μ using the minimal polynomial of the Γ_μ. For if we take

$$d(p) = m + \Gamma\,.\,p + \frac{(\Gamma\,.\,p)^2 - p^2}{m} + \frac{(\Gamma\,.\,p)^2 - p^2}{m^2}\Gamma\,.\,p + \ldots + \frac{(\Gamma\,.\,p)^2 - p^2}{m^r}(\Gamma\,.\,p)^{r-1} \tag{2.5}$$

and operate on (1.1) we get

$$\left(p^2 - m^2 + \frac{[(\Gamma\,.\,p)^2 - p^2]}{m^r}\,.\,(\Gamma\,.\,p)^r\right)\psi = 0 \tag{2.6}$$

which is the Klein–Gordon equation if

$$[(\Gamma\,.\,p)^2 - p^2](\Gamma\,.\,p)^r = 0. \tag{2.7}$$

Since the components of p_μ are arbitrary and commutative this yields the usual Harish–Chandra commutation relations for free field unique mass theories:

$$\sum(\Gamma_\mu\Gamma_\nu - \delta_{\mu\nu})\Gamma_\rho \ldots \Gamma_\epsilon = 0 \tag{2.8}$$

where the summation is over all permutations of the indices. This equation is of degree $r+2$ in the Γ_μ. By putting all indices equal we obtain the minimal polynomials for each Γ_μ:

$$(\Gamma_\mu^2 - 1)\Gamma_\mu^r = 0. \tag{2.9}$$

For the interaction case, (2.3) we operate with $d(\pi)$ and obtain

$$\left(\pi^2 - m^2 + \frac{[(\Gamma\,.\,\pi)^2 - \pi^2]}{m^r}(\Gamma\,.\,\pi)^r\right)\psi = 0. \tag{2.10}$$

Since the π_μ are not commutative, the free field relations (2.7) do not ensure that (2.9) is the Klein–Gordon equation, because

$$P_r(\pi) = [(\Gamma\,.\,\pi)^2 - \pi^2](\Gamma\,.\,\pi)^r \tag{2.11}$$

will no longer vanish as a consequence of (2.7). As Velo and Zwanzinger (1971) observe, (2.11) is of order r in the derivatives by virtue of (2.8), so

$$P_r(\pi) = O(eF\pi^r).$$

The propagation of (2.10) is determined by the coefficient of the derivatives of highest order and for $r \leqslant 1$ this will be π^2, the same as for the simple Klein–Gordon equation.

Thus for $r \leqslant 1$ the theory is bound to be causal in the presence of an external field. However, for $r > 1$, $P_r(\pi)$ will contain second-order derivatives in general and except for accidental cancellations may lead to acausality or even destroy the hyperbolic nature of the equation. Now it is well known that even in the free field case a quantizable theory (correct positivity conditions on energy and charge) based on (1.1) must have a minimal polynomial for Γ_4 with $r > 1$, i.e. Γ_4 is non-diagonalizable (Gel'fand *et al* 1963). So with high-spin theories we can expect causality problems. Amar and Dozzio (1975) in fact have shown that all theories, with or without mass spectra, are causal provided the singular subspace of Γ_4 is diagonalizable, which is the generalization of the treatment of Velo and Zwanzinger (1971). Amar and Dozzio also state that so far they have not been able to find a theory with $r > 1$ and causal propagation. In fact Capri and Shamaly (1973) have obtained a causally propagating theory for which $r = 3$. So we cannot hope to say that such theories do not exist for $r > 1$, but since the Capri–Shamaly theory is spin-1, it does not give us a higher-spin causal theory. However, it does exhibit the possibility of Γ algebras having $r > 1$ and causal propagation, and it may be that some of these algebras carry higher spin. Thus the 'accidental cancellations' of Velo and Zwanzinger can and do occur, and a more careful study of $P_r(\pi)$ might be rewarding.

A sufficient condition for causal propagation in (2.10) is clearly

$$P_r(\pi) = \mathrm{O}(\pi) \tag{2.12}$$

i.e. $P_r(\pi)$ is at most first-order in derivatives. We may still get causal theories with $P_r(\pi) \sim \mathrm{O}(\pi^2)$ but the analysis of such theories is extremely difficult. Now, introducing the notation $M_{\mu\nu\rho\ldots\epsilon} = (\Gamma_\mu \Gamma_\nu - \delta_{\mu\nu})\Gamma_\rho \ldots \Gamma_\epsilon$ we have

$$P_r(\pi) = (\Gamma_\mu \Gamma_\nu - \delta_{\mu\nu})\Gamma_\rho \ldots \Gamma_\epsilon \pi_\mu \pi_\nu \ldots \pi_\epsilon = M_{\mu\nu\rho\ldots\epsilon}\pi_\mu\pi_\nu \ldots \pi_\epsilon.$$

We now introduce the *Young symmetrizers* Y_i and the *conjugate symmetrizers* $\tilde{Y}_i$, which are elements of the group algebra of the symmetric group of r objects, S_r. They operate on the indices of an arbitrary rth rank tensor $M_{\mu_1\mu_2\ldots\mu_r}$ to produce symmetry classes of tensors providing bases for irreducible representations of GL(N) (Boerner 1963). The Y_i are constructed from the standard tableaux of appropriate Young frames by first symmetrizing with respect to indices in the rows and then anti-symmetrizing with respect to the indices in the columns of the frame. The conjugate symmetrizers $\tilde{Y}_i$ are obtained by transposing the tableau of Y_i and first anti-symmetrizing with respect to the indices in the rows and then symmetrizing with respect to the indices in the columns. Thus, if T_i is the tableau of Y_i and P_i represents the sum of all permutations of indices in the rows, while Q_i represents the sum of all permutations of the indices in the columns, each multiplied by its parity δ_q:

$$P_i = \sum p \qquad Q_i = \sum \delta_q q$$

where p are horizontal permutations and q are vertical permutations, then

$$Y_i = Q_i P_i \qquad \tilde{Y}_i = P_i Q_i.$$

An arbitrary rth rank tensor can be expanded in terms of the rth rank symmetrizers according to

$$M_{\mu\nu\ldots\epsilon} = \frac{1}{r!}\left(\sum_F s^{(F)} \sum_{i=1}^{s(F)} Y_F^{(i)} M\right)_{\mu\nu\ldots\epsilon} \tag{2.13}$$

where $Y_F^{(i)}$ denotes the symmetrizer corresponding to the ith standard tableau of the Young frame F and $s^{(F)}$ is the number of standard tableaux for the frame F. An analogous expression exists in terms of the conjugate symmetrizers. We can now prove the following theorem.

Theorem
If $(Y_iM)_{\mu\ldots\epsilon} = (Q_iP_iM)_{\mu\ldots\epsilon} = 0$ for all i such that $(\tilde{Y}_i\pi)_{\mu\ldots\epsilon} = (P_iQ_i\pi)_{\mu\ldots\epsilon} = O(\pi^r)$, $r \geq 2$, then $P_r(\pi) = O(\pi)$.

Proof
This relies simply on noting that if Y is any symmetrizer then for any tensors F and G:

$$(YF)_{\mu\ldots\epsilon}G_{\mu\ldots\epsilon} = F_{\mu\ldots\epsilon}(\tilde{Y}G)_{\mu\ldots\epsilon}$$

since:

$$F_{\mu\ldots\epsilon}(\tilde{Y}G)_{\mu\ldots\epsilon} = F_{\mu\ldots\epsilon}\sum_{pq}\delta_q\, G_{pq(\mu\ldots\epsilon)} = \sum_{pq}\delta_q\, F_{(pq)^{-1}\mu\ldots\epsilon}\, G_{\mu\ldots\epsilon}$$

by change of dummy indices, we get

$$F_{\mu\ldots\epsilon}(\tilde{Y}G)_{\mu\ldots\epsilon} = \sum_{pq}\delta_q F_{q^{-1}p^{-1}(\mu\ldots\epsilon)}G_{\mu\ldots\epsilon} = \sum_{pq}\delta_q F_{qp(\mu\ldots\epsilon)}G_{\mu\ldots\epsilon}$$

since the $q^{-1}(p^{-1})$ range over the same permutations as do the $q(p)$, we have

$$F_{\mu\ldots\epsilon}(\tilde{Y}G)_{\mu\ldots\epsilon} = (QPF)_{\mu\ldots\epsilon}G_{\mu\ldots\epsilon} = (YF)_{\mu\ldots\epsilon}G_{\mu\ldots\epsilon}.$$

Using this result we have, if S is the set of indices i such that $(\tilde{Y}_i\pi)_{\mu\ldots\epsilon} = O(\pi^r) r \geq 2$:

$$P_r(\pi) = M_{\mu\ldots\epsilon}\pi_{\mu\ldots\epsilon} = \sum_i a_i(Y_iM)_{\mu\ldots\epsilon}\pi_{\mu\ldots\epsilon}$$

on expanding the tensor $M_{\mu\ldots\epsilon}$ in terms of the Young symmetrizers

$$= \sum_{i\notin S} a_i(Y_iM)_{\mu\ldots\epsilon}\pi_{\mu\ldots\epsilon} = \sum_{i\notin S} a_iM_{\mu\ldots\epsilon}(\tilde{Y}_i\pi)_{\mu\ldots\epsilon} = O(\pi).$$

This theorem provides a simple procedure for obtaining $\tilde{U}(\Gamma)$ for some causal theories directly in Young symmetrizer form. We look at all $(\tilde{Y}_i\pi)_{\mu\ldots\epsilon}$ ranging over all standard tableaux of rank $r+2$ and, using (2.2), find all those which are second- or higher-order in the derivatives. For each such tableau we have the algebraic condition

$$(Y_iM)_{\mu\ldots\epsilon} = 0$$

which is a polynomial of degree $r+2$ in the Γ_μ. The commutation relation of Harish-Chandra (2.8) corresponds to the vanishing of the completely symmetric symmetrizer constructed from the Young frame consisting of a single row of $r+2$ boxes. The other standard tableaux yield further algebraic relations independent of this and further restricting the algebra to yield a causal theory.

3. Examples

3.1. $r = 1$

This example only serves as a simple illustration of the method since, as we have already noted, for $r = 1$ the theory is bound to be causal.

The standard tableaux of rank 3 are

$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$ $\quad (YF)_{\mu\nu\rho}=(\tilde{Y}F)_{\mu\nu\rho}$ and

$$(\tilde{Y}\pi)_{\mu\nu\rho}=\pi_{\mu\nu\rho}+\pi_{\mu\rho\nu}+\pi_{\rho\mu\nu}+\pi_{\rho\nu\mu}+\pi_{\nu\mu\rho}+\pi_{\nu\rho\mu}=O(\pi^3)$$

$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ $\quad (YF)_{\mu\nu\rho}=F_{\mu\nu\rho}+F_{\nu\mu\rho}-F_{\rho\nu\mu}-F_{\nu\rho\mu}$

$$(\tilde{Y}\pi)_{\mu\nu\rho}=\pi_{\mu\nu\rho}-\pi_{\rho\nu\mu}+\pi_{\nu\mu\rho}-\pi_{\rho\mu\nu}=O(\pi)$$

$\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ $\quad (YF)_{\mu\nu\rho}=F_{\mu\nu\rho}+F_{\rho\nu\mu}-F_{\nu\mu\rho}-F_{\rho\mu\nu}$

$$(\tilde{Y}\pi)_{\mu\nu\rho}=\pi_{\mu\nu\rho}-\pi_{\nu\mu\rho}+\pi_{\rho\nu\mu}-\pi_{\nu\rho\mu}=O(\pi)$$

$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$ $\quad (YF)_{\mu\nu\rho}=(\tilde{Y}F)_{\mu\nu\rho}$ and

$$(\tilde{Y}\pi)_{\mu\nu\rho}=\pi_{\mu\nu\rho}-\pi_{\mu\rho\nu}+\pi_{\rho\mu\nu}-\pi_{\rho\nu\mu}+\pi_{\nu\rho\mu}-\pi_{\nu\mu\rho}=O(\pi).$$

Since only the completely symmetric rank 3 symmetrizer of the π_μ is not $O(\pi)$ we need only demand that

$$(Y_{\square\square\square}M)_{\mu\nu\rho}=0$$

which gives

$$\Gamma_\mu\Gamma_\nu\Gamma_\rho+\Gamma_\mu\Gamma_\rho\Gamma_\nu+\Gamma_\rho\Gamma_\mu\Gamma_\nu+\Gamma_\rho\Gamma_\nu\Gamma_\mu+\Gamma_\nu\Gamma_\mu\Gamma_\rho+\Gamma_\nu\Gamma_\rho\Gamma_\mu=2\delta_{\mu\nu}\Gamma_\rho+2\delta_{\mu\rho}\Gamma_\nu+2\delta_{\rho\nu}\Gamma_\mu. \tag{3.1}$$

This relation is a sufficient condition for causality. As is well known this does not generate a finite algebra, and to ensure this we can append other relations of the form $(YM)_{\mu\nu\rho}=0$. The Duffin–Kemmer theory corresponds to the algebra

$$(Y_{\square\square\square}M)_{\mu\nu\rho}=0 \tag{3.2}$$

$$(Y_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}}M)_{\mu\nu\rho}=0 \tag{3.3}$$

which has two non-trivial representations and is usually expressed in the more succinct form

$$\Gamma_\mu\Gamma_\nu\Gamma_\rho+\Gamma_\rho\Gamma_\nu\Gamma_\mu=\delta_{\mu\nu}\Gamma_\rho+\delta_{\rho\nu}\Gamma_\mu$$

which is easily seen to be equivalent to (3.2) and (3.3).

3.2. $r=2$

The fourth rank standard tableaux are

$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}$ $\quad (\tilde{Y}_{\square\square\square\square}\,\pi)_{\mu\nu\rho\alpha}=O(\pi^4)$

$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$ $\quad (\tilde{Y}^{(i)}_{\begin{array}{|c|c|c|}\hline &&\\\hline \\\cline{1-1}\end{array}}\,\pi)_{\mu\nu\rho\alpha}=O(\pi^2) \qquad i=1,2,3$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\qquad (Y^{(i)}_{[2,2]}\,\pi)_{\mu\nu\rho\alpha}=\mathrm{O}(\pi)\qquad i=1,2$$

$$\begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{4}\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}\quad\begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{3}\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}\quad\begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{2}\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\qquad (\tilde{Y}^{(i)}_{[2,1,1]}\,\pi)_{\mu\nu\rho\alpha}=\mathrm{O}(\pi^2)\qquad i=1,2,3$$

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\qquad (\tilde{Y}_{[1,1,1,1]}\,\pi)_{\mu\nu\rho\alpha}=\mathrm{O}(\pi^0).$$

Using these results and the theorem of § 2 we obtain the sufficient conditions for causality in the following form:

$$\begin{aligned}&(Y_{[4]}\,M)_{\mu\nu\rho\alpha}=0\\&(Y^{(i)}_{[3,1]}\,M)_{\mu\nu\rho\alpha}=0\\&(Y^{(i)}_{[2,1,1]}\,M)_{\mu\nu\rho\alpha}=0.\end{aligned}\tag{3.4}$$

This algebra, if it possesses any non-trivial representations may provide a causal higher-spin theory.

We can continue this process and obtain algebras for higher values of r—with increasing amount of labour. It is not clear whether the $r=3$ example of Capri and Shamaly (1973) would be obtained by this process or whether it corresponds to a case in which $P_r(\pi)=\mathrm{O}(\pi^2)$.

4. Discussion of the algebra (3.4)

The general problem of analysing algebras defined by Young symmetrizer equations such as (3.4) seems to be very difficult, and requires some practical development. However, it is possible to deduce from equations such as (3.4) the number of independent elements in the algebra and, in particular, to show that the algebra generated by (3.4) is infinite.

Consider a fourth rank product $\Gamma_{\mu\nu\rho\alpha}=\Gamma_\mu\Gamma_\nu\Gamma_\rho\Gamma_\alpha$ of the Γ_μ. According to (2.13) we can expand this in the form

$$F_{\mu\nu\rho\alpha}=\frac{1}{4!}\Big(Y_{[4]}+3\sum_{i=1}^{3}Y^{(i)}_{[3,1]}+2\sum_{i=1}^{2}Y^{(i)}_{[2,2]}+3\sum_{i=1}^{3}Y^{(i)}_{[2,1,1]}+Y_{[1,1,1,1]}\Big)\Gamma_{\mu\nu\rho\alpha}.$$

But from the algebra (3.4) this must reduce to:

$$\Gamma_{\mu\nu\rho\alpha}=\frac{1}{4!}\Big(2\sum_{i=1}^{2}Y^{(i)}_{[2,2]}+Y_{[1,1,1,1]}\Big)\Gamma_{\mu\nu\rho\alpha}+\Gamma\text{ products of 2nd rank.}$$

Since we are interested in the independent products of rank four and above we can ignore the products of rank two in the following. Now let s be any permutation of S_4, operating on the indices $\mu\nu\rho\alpha$, and consider

$$s\Gamma_{\mu\nu\rho\alpha}=s\left(\frac{2}{4!}\sum_{i=1}^{2}Y^{(i)}_{[2,2]}+\frac{1}{4!}Y_{[1,1,1,1]}\right)\Gamma_{\mu\nu\rho\alpha}.$$

It can be shown that the only permutations s which satisfy

$$s\Gamma_{\mu\nu\rho\alpha}=c\Gamma_{\mu\nu\rho\alpha}$$

are those of the group

$$G=\{e,(\mu\nu)(\rho\alpha),(\mu\rho)(\nu\alpha),(\mu\alpha)(\nu\rho)\}$$

and in each case $c=1$, so that the permutations of G constitute the full symmetry imposed on the fourth rank products by the algebra (3.4)—G leaves $\Gamma_{\mu\nu\rho\alpha}$ invariant, apart from possibly the addition of second rank products.

Now if we consider an rth rank product $(r>4)$ $\Gamma_{\mu\nu\ldots\epsilon}=\Gamma_\mu\Gamma_\nu\ldots\Gamma_\epsilon$ then the algebra (3.4) allows us to perform the elements of G on any four adjacent indices—and only the elements of G—leaving $\Gamma_{\mu\nu\ldots\epsilon}$ invariant except possibly for the addition of lower rank products of the Γ_μ, which can be ignored when considering the independent rth rank products. $\Gamma_{\mu\nu\ldots\epsilon}$ is therefore effectively invariant under the set of all permutations of G on four adjacent indices, and this set generates a subgroup P of S_r which defines the complete symmetry of an rth rank product imposed by the algebra (3.4). Once the symmetry of a tensor is known, we can calculate the number of independent components, and for $\Gamma_{\mu\nu\ldots\epsilon}$ this gives the number of independent rth rank products in the algebra. Repeating the process for each r gives the number of independent elements in the algebra. If the number of components for a certain rank r is zero then the algebra is finite.

Suppose $\Gamma_{\mu\nu\ldots\epsilon}$ has symmetry

$$P=\{p_i\}$$

where p_i are permutations of S_r such that

$$p_i\,.\,\Gamma_{\mu\nu\ldots\epsilon}=\pm\Gamma_{\mu\nu\ldots\epsilon}$$

so that $\Gamma_{\mu\nu\ldots\epsilon}$ is either invariant under a permutation of P or changes sign. Then, if this defines the complete symmetry of $\Gamma_{\mu\nu\ldots\epsilon}$, the number of independent components of $\Gamma_{\mu\nu\ldots\epsilon}$ is given by

$$n_p=\frac{1}{|P|}\sum_{p_i}\pm\chi(E^a)\chi(E^b)\ldots\chi(E^f)$$

where the cycle structure of p_i is $(a,b,\ldots,f)$, $|P|=$ order of P, and $\pm$ is assigned according to whether $\Gamma_{\mu\nu\ldots\epsilon}$ is invariant or changes sign under p_i. If $\Gamma_{\mu\nu\ldots\epsilon}$ is a tensor with respect to GL(N) then $\chi(E)$ is the character of the unit element E in the vector representation, i.e. $\chi(E)=N$. In our case $\chi(E)=4$ (Bhagavantam 1966). In particular, if $\Gamma_{\mu\nu\ldots\epsilon}$ does not change sign under any element of P the n_p can never be zero and so the algebra will be infinite. This occurs in the case of the algebra (3.4)—no permutation changes the sign of $\Gamma_{\mu\nu\ldots\epsilon}$, so there will always be independent components for all ranks of products, and the algebra is therefore infinite.

We are thus at liberty to supplement the algebra (3.4) with further tensor relations to make it finite and possibly obtain a good causal theory for higher spin. The additional equation can only be obtained by equating one or more of

$$(Y^{(i)}_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}} M)_{\mu\nu\rho\alpha} \qquad \text{or} \qquad (Y_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} M)_{\mu\nu\rho\alpha}$$

to zero and in fact it seems that say

$$(Y^{(1)}_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}} M)_{\mu\nu\rho\alpha} = 0$$

will make the algebra finite. A detailed study of this algebra is in progress. It should be noted that it is not equivalent to any known causal theory, since the minimal polynomial of F_0 is $\Gamma_0^2(\Gamma_0^2 - 1)$ and the only well known theory with this minimal polynomial is the Rarita–Schwinger spin-$\frac{3}{2}$, which is acausal.

Note added in proof. The algebra is finite, but cannot yield a theory with spin greater than one. Details to be published shortly.

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